

## MACROSCOPES: MODELS FOR COLLECTIVE DECISION MAKING

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**ABSTRACT**

We introduce a new model of collective decision making, when a global decision needs to be made but the parties only possess partial information, and are unwilling (or unable) to first create a global composite of their local views. Our macroscope model captures two key features of many real-world problems: allotment structure (how access to local information is apportioned between parties, including overlaps between the parties) and the possible presence of meta-information (what each party knows about the allotment structure of the overall problem). Using the framework of communication complexity, we formalize the efficient solution of a macroscope. We present general results about the macroscope model, and also results that abstract the essential computational operations underpinning practical applications, including in financial markets and decentralized sensor networks. We illustrate the computational problem inherent in real-world collective decision making processes using results for specific functions, involving detecting a change in state (constant and step functions), and computing statistical properties (the mean).

**INTRODUCTION**

We consider collective decision making processes such as a market that acts as a central mechanism for coordinating the actions of autonomous participants. We address the questions: how does one measure the quality of the collective decision making process, and how weak can the central market mechanism be? In many applications, there is significant interest in decentralizing computation while still being able to arrive at results that cannot be computed entirely locally. We use a simple model to capture the informational complexity of computing global functions by aggregating results from participants<sup>1</sup> who are endowed with arbitrary allotments of local information. This allows us to draw conclusions about the requirements on allotments and protocols, for efficient collective information processing. A key aspect of our model is the specification of meta-information based on distinguishing perfect information, single-blind arrangements, and double-blind arrangements. Our technical framework is built on the notions of communication complexity. We assume that participants possess information which is not available to other participants; we call this the *private* information.

<sup>1</sup>We use the common English words agent, party, participant, and player synonymously, ignoring more specific usage.

This work is motivated by several applications.

A rather timely application is found in the domain of participants in electronic markets. Often, such as in financial markets, participating agents would benefit from an understanding of the global system dynamics (Darley & Outkin 2007).<sup>2</sup> For instance, agents might like to have signals that indicate the presence of herding, bubbles and other aggregate phenomena. Typically, the local view of a single agent does not provide sufficient information to reliably detect this. Moreover, in such a domain, one is tightly constrained by what information can be revealed, incentives to reveal this information, and other aspects related to privacy in computation. If we seek efficient decentralized information processing mechanisms under these constraints, then we would like to be able to determine what is or is not possible, employing only a coarse characterization of resources and endowment of information. Recent studies such as in anonymized financial chat rooms (Lu & Mizrach 2011) provide interesting insights into the behaviour of such collectives, such as the characterization of equilibria in which a subset of traders profit from the information of others. This is but one example of a larger body of economic literature related to phenomena in networked markets (Hurwicz & Reiter 2006). However, in that literature, it is not typical to investigate our question of how the allotment structure and communication protocols relate to the efficiency with which specific types of computation are achieved. For instance, change detection (Basseville & Nikiforov 1993) is of fundamental importance in financial markets – how weak a protocol is sufficient to decide a change has occurred? Recent work on the topic of complexity of financial computations by Arora, Barak, Brunnermeier & Ge (2011) indicates that this is a fertile direction to pursue.

Similar issues arise in many other application domains, such as mobile sensor networks and distributed robotics. Leonard, Paley, Lekien, Sepulchre, Fratantoni & Davis (2007) describe a mobile sensor network for optimal data gathering, using a combination of underwater and surface level sensing robots to optimally gather information such as chemical and thermal dynamics in a large

<sup>2</sup>See <http://www.bankofengland.co.uk/publications/speeches/2009/speech386.pdf> for a discussion of this issue from the perspective of financial regulation, and <http://www.bis.gov.uk/foresight/our-work/projects/current-projects/computer-trading> for information on a major study by the UK government, under the Foresight Project.

volume of water (typically measured in square miles). Similar systems have been utilized for tracking oil spills and algal blooms. A key computational method utilized by such distributed robotic networks involves distributed optimization (Bullo, Cortés & Martínez 2009). The deployment of modules in such a network needs to satisfy a spatial coverage requirement, described as a cost function, so that each module plans trajectories to optimize this criterion. The sensor fusion problem, to determine a combined estimate of an uncertain parameter, may also be posed as an optimization problem in the sense of maximizing information gain. Despite this rigorous approach, relatively little is known about how to compare different formulations of these optimization problems – given that we are interested in a certain type of global function (say, number of peaks in a chemical concentration profile or some distributional aspect of the overall field) using weak local sensing and the ability to move sensing nodes, how does one compare or otherwise characterize protocols and other aspects of the problem formulation?

A line of work that begins to touch upon some of these questions is that of Ghrist and collaborators. Baryshnikov & Ghrist (2009) use tools from algebraic topology to solve the problem of counting targets using very weak local information (such as unlabelled counts in a local neighbourhood). de Silva & Ghrist (2007) present an approach to detection of holes in coverage through decentralized computation of homology. Here again, the focus being on aspects of the specific function being computed, the authors do not address the relationship between the protocols and problem formulation, and the efficiency of computation.

Extending the idea of decentralized computation in social systems, consider the problem faced by a program committee, such as one that might review this paper. We seek a decentralized computation of a ranking problem. Similar ranking problems also occur in executive decision making such as the hiring decision in academic departments. The key issue here is that of parsimonious information sharing, coupled implicitly or explicitly with the meta-information problem. These challenges arise due to limitations on the capacities of the decision makers to exchange information with each other.

The common theme underlying all of these applications is the computation of a function based on allotment of portions of the information to parties who have reasonable amounts of computational resource but would like to keep exchange of information limited. We wish to understand how weak the corresponding protocols can be, for various types of functions. Major categories of functions of interest include change detection and ranking. We model change detection by an abstract version of the key underlying problem, of determining whether the data forms a constant or a step function. The main statistical property we consider here is the computation of the mean.

We are interested in understanding just how much communication must occur to answer various questions of interest. Therefore, instead of working with detailed models for the questions about market behaviour or sensor networks discussed previously, we have deliberately kept the models we study as *simple* as possible. *This makes our lower bound results stronger.* Determining an answer to any more realistic question will require even more information to be exchanged than in these simple models, as long as the more realistic model includes the simpler problem at its core. It therefore makes sense in our setting to study the simplest possible embodiment of each of the core problems. For the upper bounds, our results are a first step and will need to be extended to more realistic models.

## MODEL

Our model is based on the notion of *communication complexity* (Kushilevitz & Nisan 1997), which has been highly influential in computer science. A *Boolean* function models yes/no decisions, by requiring that the function take either the value 0 or the value 1. A quantity with value either 0 or 1 is known as a *bit*, and quantities that are drawn from a larger range of values can be expressed by using multiple bits; a function that is defined over a domain containing  $2^n$  different values is said to have an  $n$ -bit *input*. Say two players Alice and Bob wish to compute a Boolean function  $f$  on a  $2n$ -bit input, but Alice only has access to the first  $n$  bits and Bob to the other  $n$  bits. Alice and Bob are not *computationally* constrained, but they are *informationally* constrained. The question now is: how many bits of information do Alice and Bob need to exchange to compute  $f$  on a given  $2n$ -bit input? A *protocol* for this problem specifies, given the inputs to the players and the communication so far, which player is the next to send information, as well as what information is actually sent. There is a trivial protocol where Alice merely sends her part of the input to Bob. Bob now has all the information he needs to compute  $f$ , and he sends back the 1-bit answer to Alice. The *cost* of a protocol is the total number of bits that are exchanged; this simple protocol has a cost of  $n + 1$  for *any* function. The field of two-party communication complexity studies, for various functions  $f$  of interest, whether more efficient protocols exist. As an example, for the Equality function which tests whether Alice and Bob's inputs are exactly the same, it is known that the communication upper bound of  $n + 1$  is tight for *deterministic* protocols, but there is an improved protocol with cost  $O(\log(n))$  when the players' messages are allowed to be randomized and it is sufficient for the final answer to be correct with high probability.

The notion of communication complexity can be generalized from the two-party setting to the *multi-party* setting (Chandra, Furst & Lipton 1983). Here the number of players is not limited to two, each player has some information about the global input, and they wish to compute some Boolean function of the global input. There are two

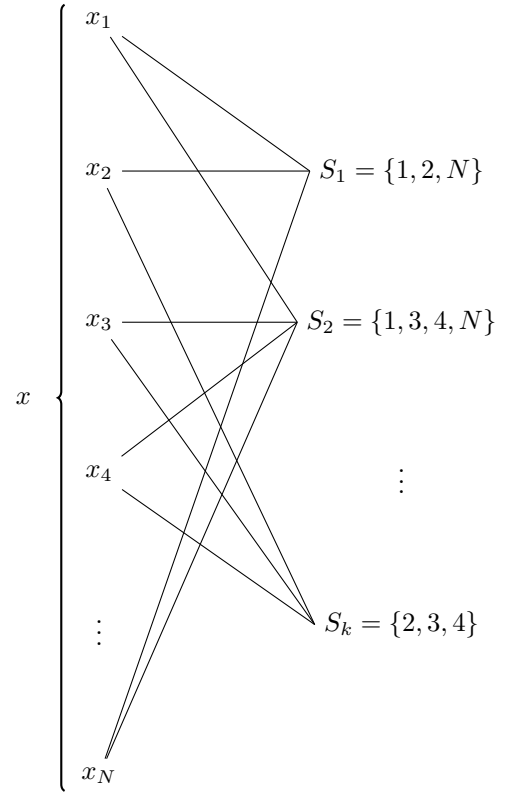
standard models for how the input is distributed among the players: the number-in-hand model (NIH) and the number-on-forehead (NOF) model. Suppose there are  $k$  players and the global input is  $N$  bits long. In the NIH model, there is some fixed partition of the global input into  $k$  parts, and each player gets one of these parts. In the NOF model, again there is a fixed partition into  $k$  parts, but the  $i$ -th player gets all the parts *except* the  $i$ -th part.

The main motivation for our model is that in many situations, such as financial markets or sensor networks, information is distributed among players in a more complex fashion than in the NOH or NIF models. Moreover, the players might not have control over which pieces of information they have access to – the *allotment* of inputs to players might be arbitrary, perhaps even adversarial. As an example, creators of financial instruments may decide which assets to bundle into pools that are then offered for sale. Purchasers of such instruments might wish to check fairness of allocation, without revealing to each other their precise holdings; or regulators might wish to check that sellers behaved impartially but without relying on full disclosure.

Yet, the players might still wish to compute some function of their global input in this less structured setting. Now different kinds of question arise than in the standard communication complexity setting. For a given function, which kinds of allotment structures allow for protocols? Does the meta-knowledge of what the allotment structure actually is make a difference to whether there is an efficient protocol or not? These questions are interesting even for simple functions which have been thoroughly investigated in the standard setting.

To be more formal, let  $f$  be a function which  $k$  players wish to compute on a global input  $x$  of size  $N$  bits. An  $(N, k)$  allotment structure is a sequence of  $k$  subsets  $S_1, S_2 \dots S_k$  of  $[N] = \{1, 2, \dots, N\}$ . An *allotment structure* corresponds to an allotment of input bits among players in the following way: Player  $i$  receives all bits  $x_j$  for  $j \in S_i$ . Note that unlike in the NOH and NIF models, this allotment of input bits is completely general – it might be the case that two players receive the same set of bits, for example. This is the main novelty of our approach. Our intention here is to model two kinds of situations. In the first, the players have little control over which pieces of information they can access – they have to do the best they can, with the available information. In the second, the allotment is made by a centralized authority, and it is of interest to study *which* allotment would most facilitate the computation in question.

A  $k$ -player *macroscope* on  $N$  bits is simply a function  $f$  on  $N$  input bits together with an  $(N, k)$  allotment structure. We will abuse notation and sometimes use a macroscope to refer to a *sequence* of functions  $f_N, N = 1 \dots \infty$ , where each function  $f_N$  depends on  $N$  bits. This will enable us to pose and study the question of the asymptotic efficiency of protocols for macrosopes.



**Figure 1. Example  $(N, k)$  allotment structure.** We wish to compute  $f(x)$  with a  $k$ -player protocol, in two situations. (Single-blind) Player  $i$  knows  $\{S_1, \dots, S_N\}$  and  $x_j$  for  $j \in S_i$ . (Double-blind) Player  $i$  only knows  $x_j$  for  $j \in S_i$ .

The generalized modelling of the allotment of inputs raises the issue of *meta-information* – how much do players know about the allotment of inputs, and how can they take advantage of this? In the case of the NIH and NOF models, the allotment is implicitly known to all players because it is fixed in advance. However, in our setting, there are two different kinds of situations – the *single-blind* situation and the *double-blind* situation. In a single-blind macroscope, all players know the allotment structure, however Player  $i$  does not know the values of any input bits apart from the ones whose indices are in  $S_i$ . In a double-blind macroscope, the players are more hampered in that they do not even know the allotment structure, however they do know the indices of the bits they receive.

It remains to formally define what a *protocol* is in our model. To keep things simple, we focus on *simultaneous-message* protocols, where each player broadcasts a sequence of bits to all players; this is often presented figuratively as each player writing their bit string on a universally viewable blackboard. A protocol *solves* a macroscope if each player can determine the value of the function on the global input simply by looking at its own input bits as well as the information written on the blackboard. The cost of a protocol for a macroscope is

then the total length of strings written by the players. In protocols for single-blind macrosopes, the message of Player  $i$  is a function of the values of bits whose indices are in  $S_i$  as well as of the allotment structure. For double-blind macrosopes, the message of Player  $i$  is a function only of the values of bits whose indices are in  $S_i$ .

We make a deliberate choice in our modelling to be highly general in terms of the allotment structure, and to be specific in terms of the structure of the actual communication. This is because our main aim is to understand the impact of the allotment structure on efficiency of communication. Our model can be extended to allow more degrees of freedom with regard to the communication structure. One way in which this can be done is to allow multiple-round protocols, where players communicate in turns, with the protocol specifying whose turn it is to communicate. Another is by allowing randomness – here each player is assumed to have access to a private source of randomness, on which its message can depend. A third way is to restrict communication to take place between specified pairs of players, i.e., there is an implicit topology of communication. This third approach is taken in the field of distributed algorithms (Lynch 1996), where however input allotment is not modelled in a flexible way.

We are interested in protocols which have communication as low as possible. This is desirable not just in terms of efficiency, such as meeting bandwidth constraints, but also in terms of *privacy*. In applications such as financial markets, the players would like to obtain some global knowledge without revealing their own inputs. Thus, Player  $i$  has more than one reason for not following the trivial protocol of publishing the values of all bits in  $S_i$ . The *lower* the communication, the *less* the information revealed about the values of bits held by individual players; we will rely on this link between parsimony of communication and the weakness of the coordination mechanism. Privacy requirements are modelled more explicitly in sub-areas of cryptography such as secure multi-party computation (Yao 1982, Goldreich 2004). We prefer not to model these requirements explicitly so as not to complicate our model too much.

We make no assumption about the relationship between the number of players and the number of bits in the global input. In an application such as sensor networks, there might be few players (sensors), each having a large amount of information, whereas in the financial markets application, there are typically many players each having few pieces of information. Our model deals equally well with both extremes.

A first observation is that to compute a non-trivial function over the global input, i.e. a function that depends on all the input bits, the allotment structure must satisfy the *covering* property – each index  $j \in [N]$  lies in at least one set  $S_i$  of the allotment structure. If the covering property did not hold, consider an index  $j$  which does not belong to the allotment, and an input  $X$  such

that  $f$  is *sensitive* to  $X$  at index  $j$ , meaning that  $f(X)$  is different from  $f(X_j^{flip})$ , where  $X_j^{flip}$  is  $X$  with the value of the  $j$ th bit flipped. By the non-triviality of  $f$ , such an input  $X$  must exist. Clearly any protocol outputs the same answer for  $X$  as for  $X_j^{flip}$  since  $j$  does not belong to the allotment, and hence the protocol cannot be correct. Henceforth, we automatically assume that a macroscope has the covering property.

There are no general necessary conditions on the allotment structure beyond the covering property for computation of non-trivial functions. But intuitively, the more “even” the allotment is, in the sense of each bit being allotted to the same number of players, the easier it is to compute a symmetric function of the inputs. We define an *even*  $(N, k)$  allotment structure as an allotment structure for which there is a number  $C$  such that each index  $i \in [N]$  belongs to exactly  $C$  distinct subsets  $S_i$ , and each subset  $S_i$  is of the same size. Clearly, for such an allotment structure, each set  $S_i$  is of size  $NC/k$ .

## RESULTS

Our first results address the question of what we can say in general about the cost of single-blind and double-blind macrosopes.

**THEOREM 1.** *Every single-blind macroscope on  $N$  bits has a protocol with cost  $N$ . Moreover, this bound is optimal.*

Note that the upper bound *does not* depend on the number of players. The proof of the upper bound in Theorem 1 takes advantage of the global knowledge the players have about the allotment structure.

**PROOF.** Consider a protocol in which each input bit  $X_i$  has a player “responsible” for it – Player  $j$  is responsible for input bit  $X_i$  iff  $i \in S_j$  and  $i \notin S_k$  for  $k < j$ . It follows from the covering property that each input bit has a player responsible for it. It should also be clear that at most one player is responsible for any given input bit. The protocol consists of players sending the values of all the bits they are responsible for. Since the macroscope is single-blind, each player knows who is responsible for which input bits. Hence each player can reconstruct the input from the information sent in the protocol, and therefore also compute the function on the input. The cost of the protocol is  $N$ , since each bit has exactly one player responsible for it.

To see that this bound is optimal, consider the macroscope consisting of the Parity function<sup>3</sup> together with an  $(N, N)$  allotment structure which allots each input bit to a distinct player. Suppose there is a protocol for this macroscope where one of the players does not send a message. Assume, without loss of generality, that this player holds the  $i$ -th bit. Then the Parity function cannot depend on the  $i$ -th bit, which is a contradiction.  $\square$

<sup>3</sup>The Parity function returns 1 if an odd number of the input bits are 1, and 0 otherwise.

**THEOREM 2.** *Every  $k$ -player double-blind macroscope on  $N$  bits has a protocol with cost at most  $2Nk$ .*

**PROOF.** The protocol giving the upper bound of Theorem 2 is a simple one. The  $j$ -th player sends an  $N$ -bit string specifying its allotment  $S_j$ , as well as at most  $N$  bits which specify the values of the bits whose indices are in the allotment  $S_j$ . Once this information is made public by each player, the players can each reconstruct the input and hence compute the function on that input.  $\square$

We do not know whether the bound in Theorem 2 is optimal in general, but we believe this to be the case for protocols that use only one round of communication.

Our primary focus is on studying the complexity of macroscopes for problems which arise in contexts such as electronic markets and distributed sensor networks. For each of these problems, we are interested in issues such as the optimal cost of solving a macroscope, the differential cost of meta-information (the reduction in cost when using a single-blind macroscope rather than a double-blind macroscope), and for single-blind macroscopes, the dependency of the cost on the allotment structure.

A fundamental problem in the context of electronic markets is the Constancy problem of detecting whether a given function is constant or not.<sup>4</sup> We model this problem as a Boolean function on  $[D]^N$ , which is 1 if all the inputs are equal, and 0 otherwise. This requires a slight adaptation of our model to inputs which are  $D$ -ary rather than binary, but this adaptation can be done in a natural way. We are able to characterize the cost of protocols for single-blind Constancy macroscopes *optimally* in terms of the allotment structure.

Given an  $(N, k)$  allotment structure, we define the intersection graph of the structure as follows. The graph has  $N$  vertices, and there is an edge between vertex  $i$  and vertex  $j$  for  $i, j \in [N]$  iff  $S_i \cap S_j \neq \emptyset$ .

**THEOREM 3.** *Every  $k$ -player single-blind Constancy macroscope on  $N$   $D$ -ary inputs can be solved with cost  $r(\lceil \log(D) \rceil) + k$ , where  $r$  is the number of connected components of the intersection graph of the allotment structure associated with the macroscope. Moreover, this bound is optimal to a constant factor.*

Note that the bound does not actually depend on  $N$ , the number of inputs!

**PROOF.** Since the macroscope is single-blind, each player knows the identities of all the other players in the same connected component in the intersection graph of the allotment structure. This is because these identities depend only on the allotment structure and not on the input.

<sup>4</sup>Although the actual values may be constantly changing, we use the abstract Boolean version of this problem to represent a more significant shift from some statistical notion of the normal behaviour. For instance, we might ask if the observed deviations from some base value are significant at a 5% confidence level.

The protocol is as follows: For each connected component, it is only the player with the smallest index in that connected component who sends a “long” message of  $\lceil \log(D) \rceil$  bits long, specifying a value in  $[D]$  that occurs in its portion of the input. In addition, each player sends a 1-bit message saying whether its portion of the input is constant or not. The players know that Constancy holds if each 1-bit message encodes “yes”, and in addition, the values in  $[D]$  sent in the long messages are all the same. Indeed, if the function is constant, it is clear that all the 1-bit messages are “yes”, and that the values in the long message are the same. To argue the converse, just notice that if a fixed player in a connected component of the intersection graph sees a constant value  $l \in [D]$ , and if all other players in the connected component see a constant value, it follows that all players in the connected component see the same constant value  $l$ .

We argue that this bound is optimal up to a factor of 2. We will give separate arguments that  $k$  bits of communication are required and that  $r \lceil \log(D) \rceil$  bits of communication are required. From these separate bounds, it follows that  $\max(k, r \lceil \log(D) \rceil) \geq (k + r \lceil \log(D) \rceil)/2$  bits of communication are required.

To see that  $k$  bits of communication are necessary, consider any allotment structure such that  $S_i \setminus (\cup_{j \neq i} S_j)$  is non-empty for each player  $i$ . Define a function  $f : [k] \rightarrow [N]$  such that  $f(i) \in S_i \setminus (\cup_{j \neq i} S_j)$  for each player  $i$ . Suppose there is a player  $i$  who does not send a message. Then the protocol gives the same answer on both the all 1’s input and the input that is all 1 except at  $f(i)$ , since the communication pattern is the same for both of these inputs. However the Constancy function differs on these inputs.

Next we show the  $r \lceil \log(D) \rceil$  lower bound. Consider any allotment structure whose intersection graph contains  $r$  connected components. Suppose that fewer than  $r \lceil \log(D) \rceil$  bits of communication are sufficient for solving the macroscope on this allotment structure. Then there is some connected component  $C$  of the intersection graph such that the corresponding players send less than  $\log(D)$  bits of communication in all. This implies that there are two values  $v_1, v_2 \in [D]$  such that the communication pattern of players in  $C$  is exactly the same when the players in  $C$  all receive the input  $v_1$  as when they all receive the input  $v_2$ . Now consider two inputs – input  $x$  in which all co-ordinates are the constant  $v_1$  and input  $y$  in which all co-ordinates outside  $C$  have value  $v_1$  and co-ordinates in  $C$  have value  $v_2$ . The communication pattern of the protocol is the same for  $x$  and  $y$ , however the Constancy function is true for  $x$  and false for  $y$ . This is a contradiction.  $\square$

Thus, for single-blind Constancy macroscopes, the critical property of the allotment structure is the number of connected components of the intersection graph. The fewer the number of connected components, the more efficiently the macroscope can be solved. We next study the situation for double-blind macroscopes.



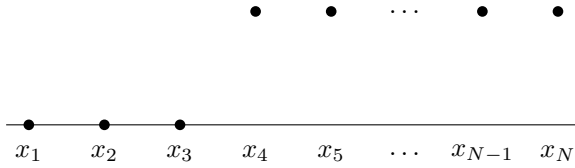
**THEOREM 4.** *Every  $k$ -player double-blind Constancy macroscope on  $N$   $D$ -ary inputs can be solved with cost  $k\lceil\log(D+1)\rceil$ . Moreover, there are  $k$ -player double-blind Constancy macrosopes which require cost  $k\lceil\log(D)\rceil$ .*

**PROOF.** The protocol giving the upper bound is simple. Each player sends a message encoding one of  $D+1$  possibilities: either the players' portion of the input is non-constant, or if it is constant, which of the  $D$  possible values it is. The protocol accepts if each message encodes the same value  $v \in [D]$ .

For the lower bound, since each player is unaware of players' allotments other than its own, the lower bound of  $r\lceil\log(D)\rceil$  in the proof of Theorem 3 holds with the maximum possible value of  $r$ , namely  $r = k$ .  $\square$

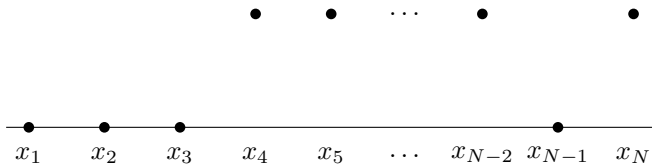
Thus, in the case of the Constancy function, the differential cost of meta-information can be quite significant, especially when the intersection graph of the allotment structure is connected.

Next we consider a formalization of the change detection problem. The Boolean Step Function (BSF) problem is defined as follows: a string  $x \in \{0,1\}^N$  evaluates to 1 if there is an index  $i$  such that  $x_j = 0$  for  $j \leq i$  and  $x_j = 1$  for  $j > i$ , or to 0 otherwise.



**Figure 2.** Boolean step function (BSF).

Possibilities for different structures of allotment lead to a twofold challenge in a BSF macroscope. First, no party may see the step, so parties need to share some information about the values they see. Second, if several parties detect a step, they then need to determine whether they are observing the same step.



**Figure 3.** A Boolean function that is not a step function.

**THEOREM 5.** *Every  $k$ -player double-blind BSF macroscope on  $N$  bits can be solved with cost  $2k\lceil\log(N)\rceil$ .*

**PROOF.** The protocol is as follows: for each  $i \in [k]$ , Player  $i$  sends two indices  $l(i)$  and  $m(i)$ , where  $l$  is the largest index in  $S_i$  for which  $x_l = 0$  and  $m$  is the smallest index in  $S_i$  for which  $x_m = 1$ . Given all these messages, each player can calculate the value of the smallest index  $m$  for which  $x_m = 1$  simply by taking the minimum of  $m(i)$  over all players  $i$ , as well as the largest index  $l$  for

which  $x_l = 0$ , simply by taking the maximum of  $l(i)$  over all players  $i$ . Note that  $BSF(x) = 0$  iff  $l = m - 1$ .  $\square$

We conjecture that the bound of Theorem 5 is tight for double-blind BSF macrosopes for protocols that use only one round of communication. For single-blind macrosopes, however, we can do better.

**THEOREM 6.** *Every  $k$ -player single-blind BSF macroscope on  $N$  bits can be solved with cost  $k\lceil\log(N)\rceil + 2k$ .*

**PROOF.** The protocol witnessing the upper bound is as follows. Each player sends a message consisting of two parts. The first part is 2 bits long, and specifies which of the following is the case: (1) the player's portion of the input is constant, (2) there is a single transition from 0 to 1 in the player's input, (3) neither (1) nor (2) holds. The second part is  $\lceil\log(N)\rceil$  bits long. The interpretation of the second part of the message is as follows: if case (1) holds for the first part, then the second part encodes which constant (either 0 or 1) the player is given. If case (2) holds, then the second part encodes the index at which a transition occurs, i.e., a number  $j \in [N]$  such that  $j \in S_i$  (assuming that the player in question is Player  $i$ ) and such that for all  $l \in S_i$ ,  $l \leq j \implies x_j = 0$  and  $l > j \implies x_j = 1$ . If case (3) holds, the contents of the second part of the message are irrelevant.

From the messages, the players can either reconstruct the input  $x$ , if case (1) or (2) holds for each player, or conclude directly that  $BSF(x) = 0$ , if case (3) holds for any player. From the input  $x$ , each player can compute  $BSF(x)$  on its own.  $\square$

In this case, too, the advantage of using single-blind macrosopes can be seen, though a matching lower bound in Theorem 5 is needed to prove this.

Next, we attempt to model the *averaging* function. Distributional statistics of different kinds need to be computed in a decentralized way in various contexts such as sensor networks. To model this, we again depart from the framework of bit strings as inputs. The input is now a sequence of real numbers  $\{x_i\}$ ,  $x_i \in [0, 1]$  for each  $1 \leq i \leq N$ . Given a parameter  $\epsilon > 0$ , we study the cost of protocols for  $\epsilon$ -Averaging macrosopes, where each player needs to arrive at an  $\epsilon$ -additive approximation to the average of the numbers  $x_i$  by using the protocol to communicate.

**THEOREM 7.** *Let  $\epsilon > 0$  be fixed. Every  $k$ -player single-blind  $\epsilon$ -Averaging macroscope on  $N$  inputs can be solved with cost  $k\lceil\log(k/\epsilon)\rceil$ .*

Notice again that there is no dependence of the cost on  $N$ , merely on the number of players and the approximation error.

**PROOF.** We again use critically the meta-information of players about the allotment structure. Each player  $i$  knows, for each index  $j \in S_i$ , the number  $N_j$  of distinct players receiving input  $x_j$ . The message sent by Player  $i$  is an  $\epsilon/k$ -additive approximation to the quantity  $\sum_{j \in S_i} x_j / N_j$ . Since the quantity is between 0 and

1, the approximation can be specified using  $\lceil \log(k/\epsilon) \rceil$  bits. Given the messages of all players, each player can compute an  $\epsilon$ -approximation to the average simply by summing all the individual approximation. Since the individual approximations are  $\epsilon/k$ -additive approximations, the sum will be an  $\epsilon$ -approximation to the average.  $\square$

In the case of Averaging macroscopes, we *can* show that the double-blind restriction is a significant one, in that it leads to a dependence of the cost on the number of players. The proof uses a dimensionality argument.

**THEOREM 8.** *Let  $\epsilon > 0$  be a parameter. There are 2-player double-blind  $\epsilon$ -Averaging macroscopes on  $N$  inputs which require cost  $N \log(1/(N\epsilon))$ .*

**PROOF.** Consider a 2-player double-blind macroscope for  $\epsilon$ -Averaging, with Player 1 receiving an allotment  $S_1 \subseteq [N]$  and Player 2 receiving an allotment  $S_2 \subseteq [N]$ . Our assumption that the allotment structure is covering implies  $S_1 \cup S_2 = [N]$ , but neither player knows anything about the other player's allotment beyond this fact.

We show that a player's message must essentially reveal its input. Consider Player 1, and let  $|S_1| = l$ . Now for each  $i \in S_1$ , consider  $S_{2,i} = [N] \setminus \{i\}$ . If Player 1 has allotment  $S_1$  and Player 2 has allotment  $S_{2,i}$ , then in a correct protocol, Player 2 eventually knows an  $\epsilon$ -approximation to  $\sum_{j \in [N]} x_j / N$ . Since it also knows all values except  $x_i$ , this implies that it knows an  $N\epsilon$ -approximation to  $x_i$ . Thus an  $N\epsilon$ -approximation to  $x_i$  should be extractable from Player 1's message. This should hold for every  $i$ , which means Player 1 must send at least  $l \log(1/(N\epsilon))$  bits. By a symmetric argument, Player 2 must send at least  $(N - l) \log(1/(N\epsilon))$  bits, which means that at least  $N \log(1/(N\epsilon))$  bits are communicated in all.  $\square$

Theorem 8 has the disadvantage that it does not say much about  $\epsilon$ , which may be large in comparison to  $N$ . However, we believe that a refinement of the proof which argues about information revealed about subsets of inputs rather than individual inputs can be used to establish an improved lower bound.

## DISCUSSION

The previous two sections have focused on the details of a model for collective information processing and properties of this model. We now take a step back to discuss why these results are of relevance to the motivating examples identified in the Introduction.

Consider the problem of agents in financial markets. It is increasingly the case that, with the emergence of diverse communication methods and agents deciding at time scales ranging from microseconds to days, common knowledge isn't so commonly available in practice. This has significant implications for dynamics (Rubinstein 1989) and much of the modern discussion regarding markets is related to such issues (Arora et al. 2011). In this setting, there is a need for diagnostic

tools that could provide useful signals – by computing global properties, i.e. functions, based on local information that can be used subject to limitations on protocols. For instance, has there been a change from a 'constant' level in a global sense? Or, is there a significant difference – in the form of a step change – between segments of a networked market? We illustrate the use of concepts from complexity theory to address abstract versions of such questions. Indeed, our techniques could be used to answer further such questions – about statistical distributions, ranking queries, etc. A key novelty, even in comparison to the state of the art in communication complexity, is that we consider an arbitrary endowment of inputs and meta-information such as single and double blind protocols.

An important general direction for future work in this area would be to extend our analysis to more directly address the subtleties of the above mentioned dynamics as they occur in applications of interest. Also, we would like to better understand the relationships between our model of decentralized computation under informational constraints and previous established models, such as (Hurwicz & Reiter 2006), which employ different methods and focus more on temporally extended sequential protocols such as auctions.

In terms of more specific questions, our model could be extended in various ways. We have considered the case of a *simple* structure of communication, with simultaneous messages sent in one round of communication, and a possibly complex allotment structure of the inputs. We could allow more sophisticated communication structure. For example, double-blind protocols with two rounds of communication can emulate single-round single-blind protocols with some loss in efficiency, simply by using the first round to share publicly information about the input allotment, and then running the single-blind protocol in the second round. More generally, communication might be restricted to occur only between specific *pairs* of players. In the context of sensor networks, for instance, it is natural to model both the location of information and the structure of communication as governed by the topology of the ambient space. Even studying very simple functions such as Constancy in such general models appears to be interesting.

We emphasize that we are interested in understanding the communication requirements of even very simple functions in modelling frameworks that render them non-trivial. This distinguishes our work from the existing research on communication complexity, where functions such as Constancy and BSF are trivial because the model of communication is so simple. We are especially interested in modelling some aspects of collective information processing, such as information overlap and meta-information, which have been neglected so far and which we believe can have a significant impact on the efficiency of communication. The way we capture these notions is quite flexible and can be used both to model

computation of continuous quantities such as Average and computation of discrete quantities such as Boolean functions, as we have illustrated with our results.

Another direction that we find compelling is modelling meta-information in a more sophisticated way. As of now, we have the single-blind model and the double-blind model. But there are various intermediate notions that are reasonable to study. For example, each player might know the number of players and the number of inputs but nothing about which inputs are given to which other players. Or a player might know which other players also receive the inputs it receives, but nothing about inputs it does not receive. Or some global property of the allotment structure, such as that the allotment structure is even, might be known. Notice that in the case of Averaging macroscopes, there *is* an efficient protocol whose cost doesn't depend on the number of players if the allotment structure is even and the size of each allotment is known. The protocol simply involves the players summing all their inputs and dividing by a universal constant. In general, one can ask: assuming that there are efficient single-blind protocols known, what is the minimal information about the allotment structure required to give efficient protocols?

Ranking is an important subject that we have not addressed in this paper. Some interesting problems can be captured as ranking macroscopes, and we leave their study for further work.

We have seen that in some cases more information hinders making a global decision, rather than helping. More generally, why is allotment structure important? We are striving to fully understand how the allotment structure of information affects our ability to efficiently answer questions that require global information.

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